Consistent dictionary learning for signal declipping

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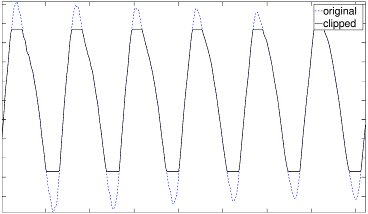
Background: declipping



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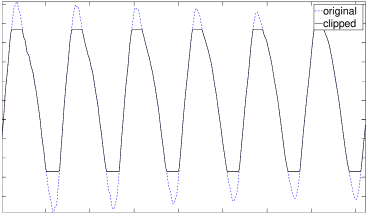

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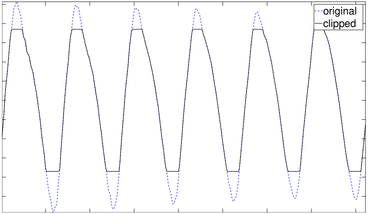

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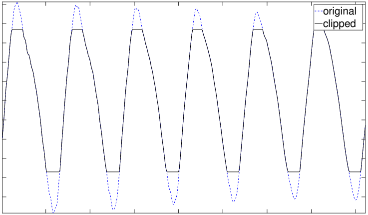
Declipping:

- Recovering original signal from clipped signal
- Non-linear, highly under-determined inverse problem (only low energy samples are available)
- Declipping strategies: AR modelling, bandwidth-limited models, Bayesian approaches

More recently: **sparsity based approaches**:



- **y**: measured clipped signal
- **x**: original clean signal

- Assume original signal is sparse $\mathbf{x} = \mathbf{D}\,\boldsymbol{\alpha}$, where $\mathbf{D} \in \mathbb{R}^{N \times M}$ $(N \leq M)$ **overcomplete dictionary** and $\|\,\boldsymbol{\alpha}\,\|_0 \leq K$.

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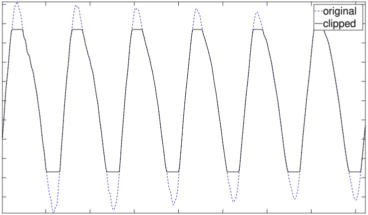

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- "Straighforward" declipping formulation:

$$\min_{\boldsymbol{\alpha}} \|\,\mathbf{M^r}(\mathbf{y} - \mathbf{D}\,\boldsymbol{\alpha})\|_2^2 \quad \text{s.t.} \quad \|\,\boldsymbol{\alpha}\,\|_0 \leq K, \tag{1}$$

where $\mathbf{M^r}$ is a binary sensing matrix defining the *reliable* (i.e. unclipped) samples.

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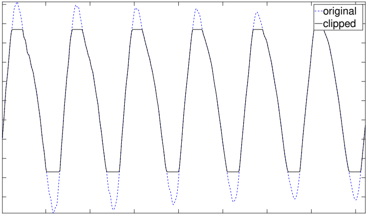

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  where $\mathbf{M^r}$ is a binary sensing matrix defining the *reliable* (i.e. unclipped) samples.
- Many well-known algorithms to solve (1), e.g. (Orthogonal) Matching Pursuit, Iterative Hard Thresholding (IHT), etc...

Example:

- $\hat{\boldsymbol{\alpha}} = \text{argmin}_{\boldsymbol{\alpha}} \| \mathbf{M^r}(\mathbf{y} - \mathbf{D}\,\boldsymbol{\alpha})\|_2^2 \quad \text{s.t.} \quad \| \boldsymbol{\alpha} \|_0 \leq K$
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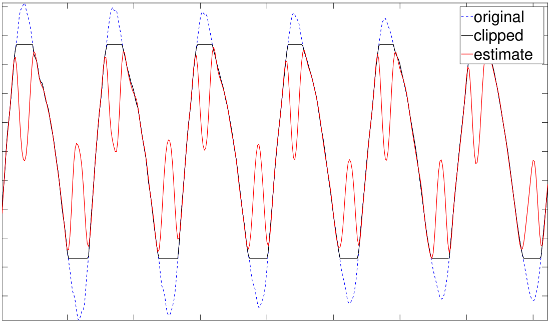

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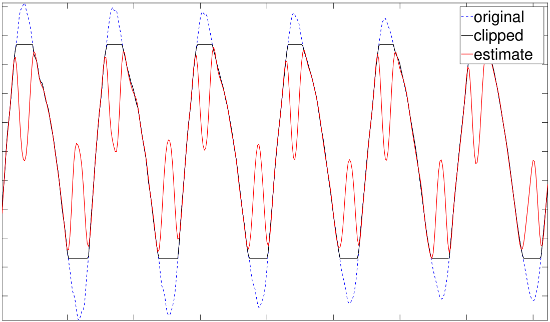

- **“classical” well known sparse recovery algorithms do not perform well on declipping!**

- Strategy: enforce reconstructed samples to be **above/below the clipping threshold** [Adler2012]:

$$\min_{\boldsymbol{\alpha}} \| \mathbf{M}^{\mathbf{r}}(\mathbf{y} - \mathbf{D}\,\boldsymbol{\alpha})\|_2^2 \quad \text{s.t.} \begin{cases} \| \boldsymbol{\alpha} \|_0 \leq K \\ \mathbf{M}^{\mathbf{c}+}\,\mathbf{D}\,\boldsymbol{\alpha} \succeq \theta^+\,\mathbf{M}^{\mathbf{c}+}\,\mathbf{1} \\ \mathbf{M}^{\mathbf{c}\text{-}}\,\mathbf{D}\,\boldsymbol{\alpha} \preceq \theta^-\,\mathbf{M}^{\mathbf{c}\text{-}}\,\mathbf{1} \end{cases} \tag{2}$$

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- Formulation is consistent with the clipping process (fully models our knowledge about the clipping process)
- **Difficult constrained, high-dimensional, non-convex optimization problem!**

ADMM-based sparse declipper: (SPADE) [Kitic,2015]

$$\min_{\boldsymbol{\alpha}} \| \boldsymbol{\alpha} \|_0 + \mathbb{1}_{\mathcal{C}(\mathbf{y})}(\mathbf{D}\,\boldsymbol{\alpha}) \tag{3}$$

with $\mathbb{1}_{\mathcal{C}(\mathbf{y})}$ indicator function of the set $\mathcal{C}(\mathbf{y})$, and:

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- **Unstable** (does not converge when sparsity level K is fixed)

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  - Computationally simple

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- Not addressed in the context of clipped data

Dictionary learning for declipping?

Dictionary learning often performs many iterations over large datasets, so we need a formulation that is:

- **computationally tractable**
- **stable**
- **does not make any assumption on the dictionary** (tightness etc...)

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Proposed problem formulation

- Reformulate declipping as a problem of **minimizing the distance between the approximated signals $\mathbf{D}\,\boldsymbol{\alpha}_t$, and their feasibility sets** $\mathcal{C}(\mathbf{y}_t)$:

$$\min_{\mathbf{D}\in\mathcal{D},\boldsymbol{\alpha}_t} \sum_t \mathrm{d}(\mathbf{D}\,\boldsymbol{\alpha}_t, \mathcal{C}(\mathbf{y}_t))^2 \quad \text{s.t.} \quad \forall t, \|\,\boldsymbol{\alpha}_t\,\|_0 \leq K, \tag{8}$$

with:

$$\mathcal{C}(\mathbf{y}_t) \triangleq \{\mathbf{x} |\, \mathbf{M}^{\mathbf{r}}\,\mathbf{y}_t = \mathbf{M}^{\mathbf{r}}\,\mathbf{x}, \mathbf{M}^{\mathbf{c}+}\,\mathbf{x} \succeq \mathbf{M}^{\mathbf{c}+}\,\mathbf{y}_t, \mathbf{M}^{\mathbf{c}\text{-}}\,\mathbf{x} \preceq \mathbf{M}^{\mathbf{c}\text{-}}\,\mathbf{y}_t\}, \tag{9}$$

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- **Enforces signals to be "close" to their feasibility sets, instead of being exactly in the set.**
- Minimize distance to a **set**, instead of minimizing distance to a point!

Properties of $\mathrm{d}(\mathbf{x}, \mathcal{C}(\mathbf{y}))^2$:

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Moreover since $\mathcal{C}(\mathbf{y})$ is convex:

- $\mathrm{d}(\mathbf{x}, \mathcal{C}(\mathbf{y}))^2$ is **convex**, as a minimum of convex functions over a convex set [Boyd,2004].

Differentiability of $d(\mathbf{x}, \mathcal{C}(\mathbf{y}))^2 = \min_{\mathbf{z} \in \mathcal{C}(\mathbf{y})} \|\mathbf{x} - \mathbf{z}\|_2^2$:

Danskin's Min-Max theorem [Bonnans,1998]:

- $\mathcal{C}$ a compact set
- $g(\mathbf{x}) = \min_{\mathbf{z} \in \mathcal{C}} \phi(\mathbf{x}, \mathbf{z})$
- $\forall \mathbf{z} \in \mathbb{R}^N$, $\phi(., \mathbf{z})$ is **differentiable** with gradient $\nabla_{\mathbf{x}} \phi(\mathbf{x}, \mathbf{z})$
- $\phi(\mathbf{x}, \mathbf{z})$ and $\nabla_{\mathbf{x}} \phi(\mathbf{x}, \mathbf{z})$ are **continuous** on $\mathbb{R}^N \times \mathbb{R}^N$

If:

- $\text{argmin}_{\mathbf{z} \in \mathcal{C}} \phi(\mathbf{x}, \mathbf{z}) = \{\mathbf{z}^*\}$ is **unique**

Then:

- **$g(.)$ is differentiable with gradient:**

$$\nabla g(\mathbf{x}) = \nabla_{\mathbf{x}} \phi(\mathbf{x}, \mathbf{z}^*). \tag{11}$$

Differentiability of $\mathrm{d}(\mathbf{x}, \mathcal{C}(\mathbf{y}))^2$:

Here:

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- $\operatorname{argmin}_{\mathbf{z} \in \mathcal{C}(\mathbf{y})} \|\mathbf{x} - \mathbf{z}\|_2^2 \triangleq \Pi_{\mathcal{C}(\mathbf{y})}(\mathbf{x})$ **orthogonal projection** of $\mathbf{x}$ onto set $\mathcal{C}(\mathbf{y})$.

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- $\Rightarrow \mathrm{d}(\mathbf{x}, \mathcal{C}(\mathbf{y}))^2$ is differentiable with gradient:

$$\nabla_{\mathbf{x}} \frac{1}{2} \mathrm{d}(\mathbf{x}, \mathcal{C}(\mathbf{y}))^2 = \mathbf{x} - \Pi_{\mathcal{C}(\mathbf{y})}(\mathbf{x}) \tag{12}$$

Summary/Comparison with Linear Least Squares:

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- **Performing consistent sparse declipping is as simple as doing (regular) sparse coding!**

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Algorithm

$$\min_{\mathbf{D}\in\mathcal{D},\boldsymbol{\alpha}_t} \sum_t \mathrm{d}(\mathbf{D}\,\boldsymbol{\alpha}_t, \mathcal{C}(\mathbf{y}_t))^2 \quad \text{s.t.} \quad \forall t, \|\,\boldsymbol{\alpha}_t\,\|_0 \leq K$$

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Alternate minimization between sparse coefficients $\boldsymbol{\alpha}_t$ and dictionary $\mathbf{D}$:

Proposed Consistent dictionary learning algorithm:

Iterate until convergence:

Sparse coding step:

for $t = 1, \ldots, T$:

$$\boldsymbol{\alpha}_t \leftarrow \boldsymbol{\alpha}_t + \mu_1 \mathbf{D}^T(\Pi_{\mathcal{C}(\mathbf{y}_t)}(\mathbf{D}\,\boldsymbol{\alpha}_t) - \mathbf{D}\,\boldsymbol{\alpha}_t) \qquad \triangleright \text{ Gradient descent step}$$

$$\boldsymbol{\alpha}_t \leftarrow \mathcal{H}_K(\boldsymbol{\alpha}_t) \qquad \triangleright \text{ Hard-thresholding}$$

Dictionary update step:

$$\mathbf{D} \leftarrow \Pi_{\mathcal{D}}\Big(\mathbf{D} + \mu_2 \sum_t (\Pi_{\mathcal{C}(\mathbf{y}_t)}(\mathbf{D}\,\boldsymbol{\alpha}_t) - \mathbf{D}\,\boldsymbol{\alpha}_t)\,\boldsymbol{\alpha}_t^T\Big) \qquad \triangleright \text{ Gradient desc.}$$

Explicit computation of projection operator

- The algorithm requires the computation of projections $\Pi_{\mathcal{C}(\mathbf{y})}(\mathbf{D}\,\boldsymbol{\alpha})$ at each iteration.

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- Equivalent to regularization-based methods
- Sparse coding step is equivalent to Consistent IHT [Kitic2013]!

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Experiments

- Tested on audio signals, $T = 2500$ time frames of size $N = 256$, and dictionaries of size $M = 512$.
- Signal-to-Distortion ratio (SDR), computed on the clipped samples, at different clipping levels

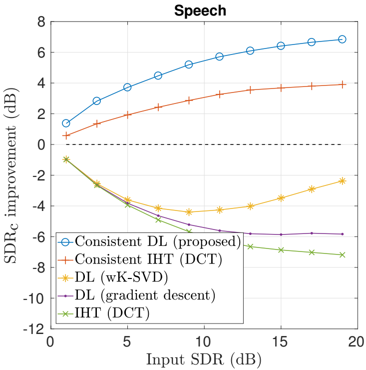

Figure: Comparison with state-of-the-art dictionary learning algorithms

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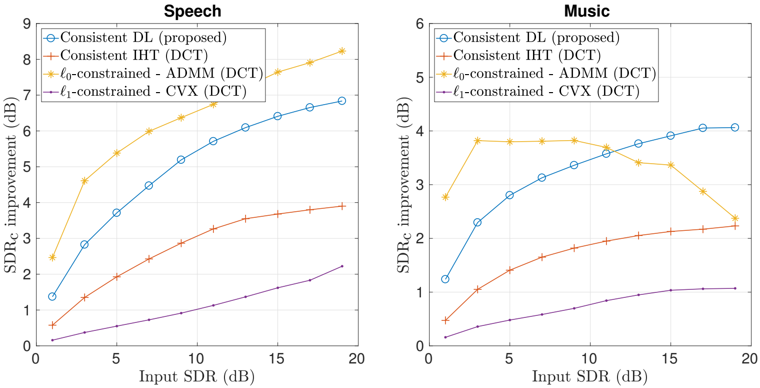

Figure: Comparison with state-of-the-art declipping algorithms

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- extend other sparse coding/dictionary learning algorithms to optimize the proposed cost function

Consistent dictionary learning for signal declipping
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Code and audio examples available at:

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MacSeNet
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