A greedy algorithm with learned statistics for sparse signal reconstruction

MacSeNet
Machine Sensing Training Network

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Abstract

We propose a greedy OMP-like algorithm that uses simple first order statistics, for sparse signal reconstruction. The proposed algorithm performs better when reconstructing a signal from a few noisy samples, with statistics learned from a training signal.

Problem formulation

We consider the sparse signal reconstruction problem:

$$\hat{x} = \underset{x}{\operatorname{argmin}} \|y - Dx\|_2^2, \text{ s.t. } \|x\|_0 < K, \tag{1}$$

where y is a **noisy signal** with **missing samples**.

Orthogonal Matching Pursuit (OMP) [1]

Starting from residual $r = y$, and support $\Omega = \emptyset$, iterate:

- **Select an atom** from the dictionary:
$$\hat{\imath} = \underset{i}{\operatorname{argmax}} \frac{|\langle d_i, r_{k-1}\rangle|^2}{\|d_i\|^2} \tag{2}$$
- **Update the coefficients** in the support Ω_k:
$$\begin{aligned} x_k &= \underset{u}{\operatorname{argmin}} \|y - D_{\Omega^k} u\|_2^2 \\ &= (D_{\Omega^k}^T D_{\Omega^k})^{-1} D_{\Omega^k}^T y \end{aligned} \tag{3}$$
- **Update the residual**:
$$r_k = y - D_{\Omega^k} x_k \tag{4}$$

Properties of OMP:

- (2), (3) and (4) ensures that **atoms are not selected twice**.
- the coefficient update (3) ensures **steepest descent** of the residual error at each iteration.

Covariance-Assisted Matching Pursuit (CAMP) [2]

[2] proposed to improve the coefficient update step by introducing the **mean** μ^{nz} and **covariance** Λ^{nz} of the *non-zero* coefficients:

- Select atom as in (2)
- Solve $y = D_{\Omega^k} x_k + v$, assuming $x_k \sim \mathcal{N}(\mu_k, \Lambda_k)$ and $v \sim \mathcal{N}(0, \Sigma)$:
$$x_k = (D_{\Omega^k}^T \Sigma^{-1} D_{\Omega^k} + \Lambda_k^{-1})^{-1} (D_{\Omega^k}^T \Sigma^{-1} y + \Lambda_k^{-1} \mu_k). \tag{5}$$
- Update the residual as in (4).

Analysis of CAMP:

- Takes into account the mean and covariance of the non-zero coefficients in the coefficient update. However:
- The new update step (5) does not guarantee multiple selection of the same atom
- (5) does not correspond to steepest decent of residual error any more.

Proposed algorithm

We propose instead to solve:

$$\hat{x} = \underset{x}{\operatorname{argmin}} [(y - Dx)^T \Sigma^{-1} (y - Dx) + (x - \mu)^T \Lambda^{-1} (x - \mu)] \text{ s.t. } \|x\|_0 < K, \tag{6}$$

with μ and Λ the **mean** and **covariance** of x. (6) can be reformulated as:

$$\hat{x} = \underset{x}{\operatorname{argmin}} \left\| \begin{bmatrix} \Sigma^{-1/2} y \\ \Lambda^{-1/2} \mu \end{bmatrix} - \begin{bmatrix} \Sigma^{-1/2} D \\ \Lambda^{-1/2} \end{bmatrix} x \right\|^2 \text{ s.t. } \|x\|_0 < K \tag{7}$$

$$= \underset{x}{\operatorname{argmin}} \|\tilde{y} - \tilde{D} x\|_2^2 \text{ s.t. } \|x\|_0 < K \tag{8}$$

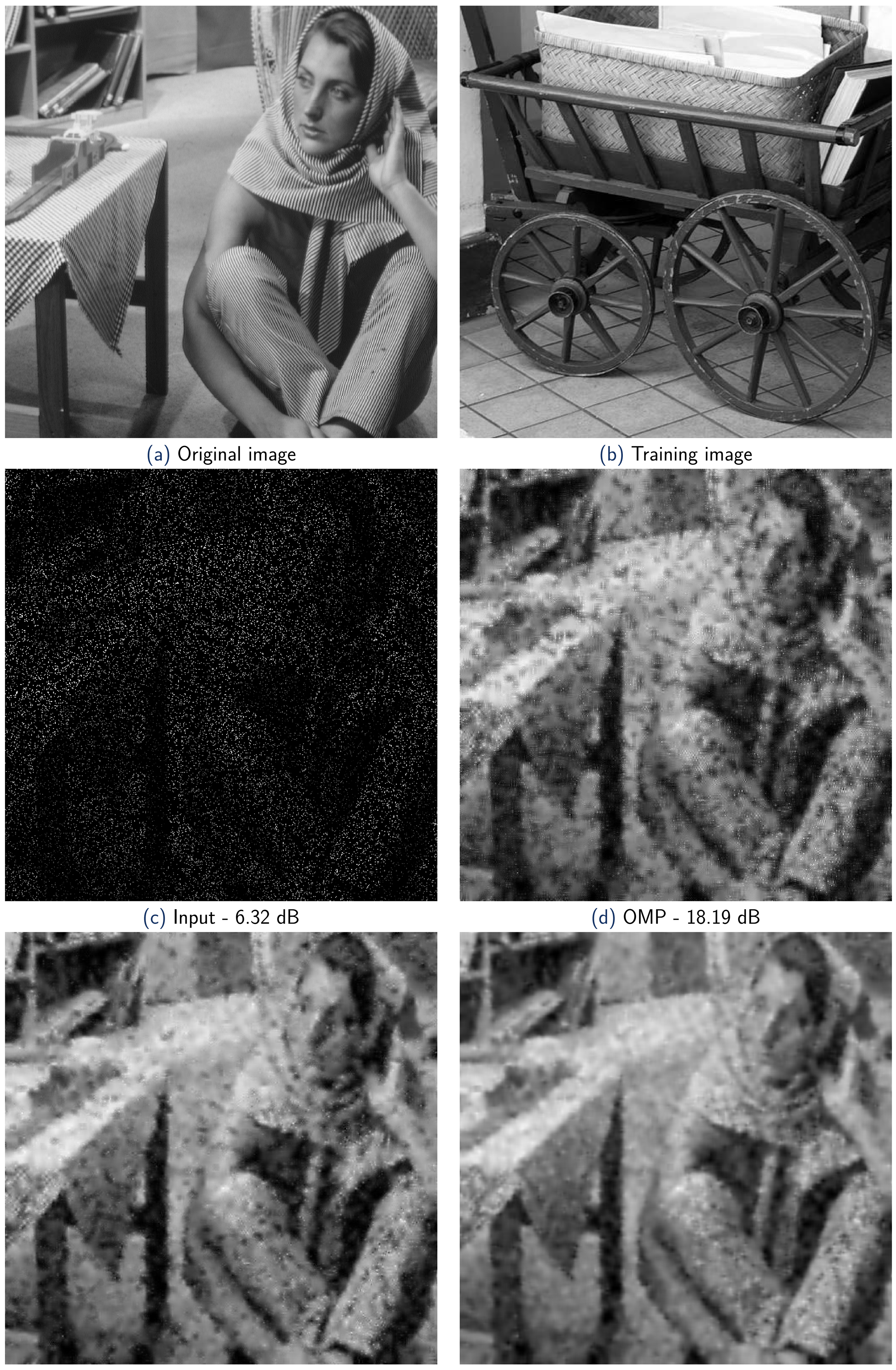
(6) can be solved in a **greedy OMP-like** way. Starting from $r_1 = y$ and $r_2 = \mu$, iterate:

- **Atom selection step**:
$$\hat{\imath} = \underset{i}{\operatorname{argmax}} \frac{|e_i^T D^T \Sigma^{-1} r_1^{k-1} + e_i^T \Lambda^{-1} r_2^{k-1}|^2}{e_i^T D^T \Sigma^{-1} D e_i + e_i^T \Lambda^{-1} e_i}, \tag{9}$$
- **Coefficient update**:
$$x_k = (D_{\Omega^k}^T \Sigma^{-1} D_{\Omega^k} + S_k \Lambda^{-1} S_k^T)^{-1} (D_{\Omega^k}^T \Sigma^{-1} y + S_k \Lambda^{-1} \mu) \tag{10}$$
- **Residual update**:
$$\begin{aligned} r_1^k &= y - D_{\Omega^k} x_k \\ r_2^k &= \mu - S_k^T x_k \end{aligned} \tag{11}$$

⇒ **Takes into account mean μ and covariance Λ at every step**
⇒ **Same practical properties as OMP**

Performance evaluation

We reconstruct an image from 10% of noisy pixels, using statistics learnt on a training image:



(a) Original image (b) Training image
(c) Input - 6.32 dB (d) OMP - 18.19 dB
(e) CAMP - 20.15 dB (f) Proposed - 21.22 dB

Figure 1: Reconstruction of image (a) with 90% of missing pixels, and additive Gaussian noise ($\sigma = 30$). The statistics (mean μ and covariance Λ) are learned from the training image (b). Each algorithm was performed using 8×8 patches, a DCT dictionary $D \in \mathbb{R}^{64 \times 256}$ and a maximum number of atoms $K_{\max} = 32$.

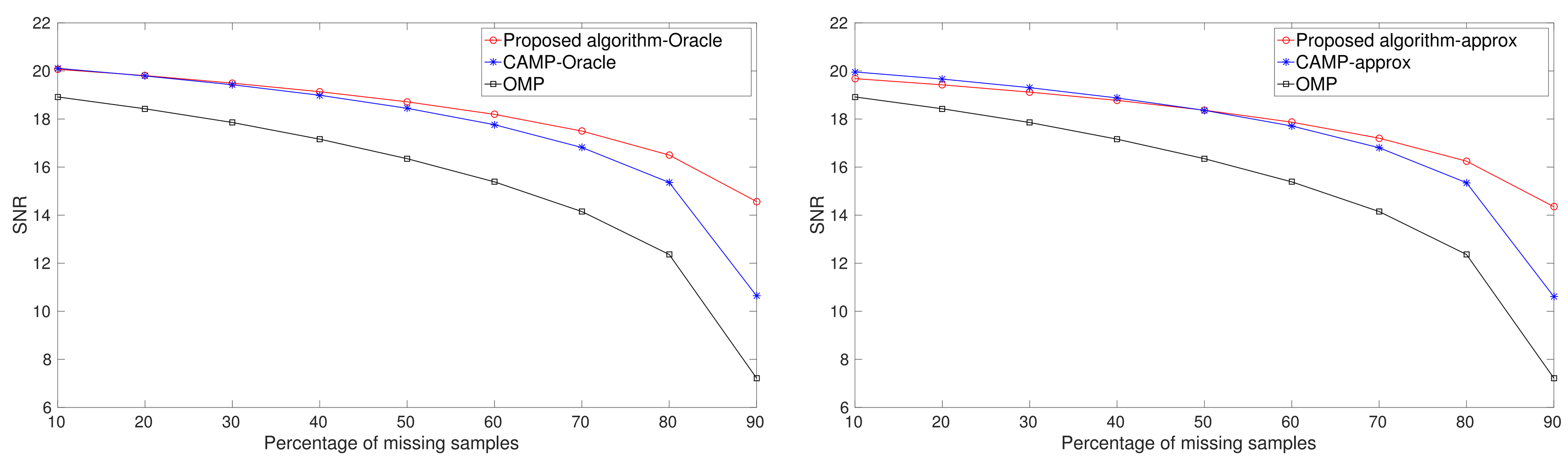


Figure 2: Comparison of OMP, CAMP and proposed algorithm on image restoration from a few noisy samples. Left: statistics learned from the test image. Right: statistics learned from a training image. The results are averaged over 150,000 patches taken from 6 images.

Conclusion

The proposed algorithm allows to take into account the first order statistics of the coefficient vector, while keeping the practicality of OMP. Experiments show improved performance when only a few noisy samples are available.

References

[1] Y. C. Pati, R. Rezaiifar, and P. S. Krishnaprasad, "Orthogonal matching pursuit: recursive function approximation with applications to wavelet decomposition," in *1993 Conference Record of The Twenty-Seventh Asilomar Conference on Signals, Systems and Computers*, pp. 40–44.

[2] A. Adler, "Covariance-Assisted Matching Pursuit," *IEEE Signal Processing Letters*, vol. 23, pp. 149–153, Jan. 2016.

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